Rotation theory of torus homeomorphisms IV

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Brouwer-Le Calvez foliations

A Brouwer-Le Calvez pair for $f\colon S \to S$ is a pair $(\mathcal{F}, (f_t)_{t\in[0,1]})$ of

- An oriented topological foliation $\mathcal{F}$ with singularities;
- An isotopy $(f_t)_{t\in[0,1]}$ from $f_0 = \mathrm{Id}$ to $f_1 = f$

such that:

- $X := \mathrm{Sing}(\mathcal{F})$ is fixed pointwise by the isotopy, and
- $\mathcal{F}$ is **dynamically transverse**: for every $z \in S\backslash X$, the arc $\gamma_z = (f_t(z))_{t\in[0,1]}$ is homotopic in $S\backslash X$ (with fixed endpoints) to an arc positively transverse to $\mathcal{F}$ (i.e. crossing leaves left to right).

Theorem [LC05, Jau14] + Beguin-Crovisier-Le Roux

S surface, f isotopic to Id, then "BLC pairs exist".

Linking numbers for open topological disks

Suppose $S = \mathbb{R}^2$, $(\mathcal{F}, (f_t)_{t\in[0,1]})$ a BLC pair, $X = \mathrm{Sing}(\mathcal{F})$.

- Let U be an open invariant topdisk.
- For $p \in U \backslash X$ we may define a linking number $L(U, p)$:
- Choose $z \in U \backslash X$. Then $\gamma_z = (f_t(z))_{t\in[0,1]}$ is not a loop, but
- choose any arc $\sigma \subset U$ joining $f(z)$ to z,
- define $L(U, p) = \mathrm{wind}(\alpha, p)$.
- Does not depend on the choice of σ or z (exercise).

Linking lemma

Every $z \in \mathbb{R}^2 \backslash X$ has a neighborhood V_z such that if some open f-invariant topdisk U **without wandering points** intersects V_z, then one of the sets $\omega(\Gamma_z)$ or $\alpha(\Gamma_z)$ consists of a single point $p \in X$ and either

(a) $p \in U$, or

(b) $L(U, p) \neq 0$,

Proof of Bounded Disks Lemma

- $f \in \mathrm{Homeo}_{0,\mu}(\mathbb{T}^2)$ and $\mathrm{Fix}(f)$ is ~~inessential~~ **totally disconnected**.
- Suppose there is an invariant topological disk U such that $\mathcal{D}(U) = \infty$.
- Choose a lift $\widetilde{U}$ of U and a lift $\widetilde{f}$ such that $\widetilde{f}(\widetilde{U}) = \widetilde{U}$.
- $((f_t), \mathcal{F})$ a BLC pair for f that lifts to a BLC pair $((\widetilde{f}_t), \widetilde{\mathcal{F}})$ for $\widetilde{f}$.
- $\pi|_{\widetilde{U}} \colon \widetilde{U} \to U$ is a homeomorphism and $\pi \widetilde{f}|_{\widetilde{U}} = f\pi|_{\widetilde{U}}$, so $\widetilde{U}$ has finite area and no wandering points.
- Same thing for $\widetilde{U} + v$, $v \in \mathbb{Z}^2$
- Choose $\{V_z : z \in \mathbb{R}^2\}$ as in the Linking Lemma. Equivariant.
- Fix a connected $Q \subset \mathbb{R}^2$ such that $[0,1]^2 \subset Q$ and $\partial Q \cap \widetilde{X} = \emptyset$.
- Cover ∂Q with finitely many $V_{z_1}, \ldots, V_{z_m}$.
- $A =$ union of ω- or α-limit sets of $\widetilde{\Gamma}_{z_i}$'s which are singletons. Finite.
- $\widetilde{U}$ has a fixed point. $\mathrm{link}(\widetilde{U}) = \{p \in X \backslash \widetilde{U} : L(\widetilde{U}, p) \neq 0\}$ compact.
- $\widetilde{U} + v$ intersects $\partial Q \implies \mathrm{link}(\widetilde{U} + v) \cap A \neq \emptyset$ or $(\widetilde{U} + v) \cap A \neq \emptyset$.
- $\mathrm{link}(\widetilde{U} + v) = \mathrm{link}(\widetilde{U}) + v$
- Blackboard.

Rotational deviations: rotation sets with empty interior

Suppose $\rho(\widetilde{f})$ is an interval.

If f has no periodic points:

- Alejandro's talks.

If f has periodic points.

- interval with irrational slope and exactly one rational endpoint;
- interval with rational slope. Reduces to $\{p/q\} \times [a, b]$;
- interesting case: $\{0\} \times [a, b]$ (consider $\widetilde{g} = \widetilde{f}^q - (p, 0)$)

Vertical rotation interval with periodic points

Theorem [GKT14]

If $\rho(\widetilde{f}) = \{0\} \times [a, b]$, then $\widetilde{f}$ has uniformly bounded horizontal displacement: $\sup_{z\in\mathbb{R}^2, n\in\mathbb{N}} |(\widetilde{f}^n(z) - z)_1| < \infty$ ("f is annular"). In fact, there exists an invariant "vertical" topological annulus.

Remark

The first part of the theorem was proved in the general setting (no area-preserving hypothesis) by Dávalos [Dáv13].

Tools

- Bounded disks lemma / strictly toral dynamics.
- ω-sets ("stable sets" of ∞ in the universal covering);
- geometric / quasiconvexity properties of eventually free chains;
- prime ends rotation numbers / boundary dynamics.

ω-sets

Notation:

- $H_t^+ = \{(x,y) : x \geqslant t\}$, $H^- = \{(x,y) : x \leqslant t\}$.
- $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ and $\widetilde{f}$ a lift.
- the set $K^+ = \bigcap_{n=-\infty}^{\infty} \widetilde{f}^n(H_t^+) \cup \{\infty\}$ is compact in $\mathbb{R}^2 \cup \{\infty\}$.
- $K_\infty^+ =$ connected component of K containing ∞.
- Define $\omega_t^+(\widetilde{f}) = K_\infty^+ \backslash \infty$.

Alternatively, denote $\mathrm{ucc}(C)$ the union of all unbounded connected components of a set C. Then

$$\omega_t^+(\widetilde{f}) = \mathrm{ucc}\Big(\bigcap_{n=-\infty}^{\infty} \widetilde{f}^n(H_t^+)\Big).$$

The set $\omega_t^-(\widetilde{f})$ is defined analogously.

Properties of ω-sets

$$\omega_t^{\pm}(\widetilde{f}) = \mathrm{ucc}\Big(\bigcap_{n=-\infty}^{\infty} \widetilde{f}^n(H_t^{\pm})\Big).$$

The sets $\omega_t^{\pm}(\widetilde{f})$ may be empty. But when they are not:

- They are closed, $\widetilde{f}$-invariant.
- Every connected component of $\omega^{\pm}$ is unbounded.
- $\mathbb{R}^2 \backslash \omega^{\pm}$ is simply connected ($\simeq \mathbb{R}^2$).
- $\omega_t^{\pm} + (0, b) = \omega_t^{\pm}$ for all $b \in \mathbb{Z}$.
- If $s > t$, then $\omega_s^+ \subset \omega_t^+$ and $\omega_t^- \subset \omega_s^-$.
- $\omega_t^{\pm} + (a, 0) = \omega_{t+a}^{\pm}$ if $a \in \mathbb{Z}$.
- So $\omega_k^+ = \omega_0 + (k, 0)$, $k \in \mathbb{Z}$ is a decreasing sequence of sets.

Lemma

$\rho(\widetilde{f}) \subset \{0\} \times \mathbb{R} \implies \omega_0^+ \neq \varnothing \neq \omega_0^-$ ($\implies \omega_t^\pm \neq \varnothing$ for all t).

$$W_n = \mathrm{ucc}\Big(\bigcap_{k=-n}^{n} \widetilde{f}^k(H_0^+)\Big), \quad \omega_0^+ = \bigcap_{n\geqslant 0} W_n$$

- If $W_n \subset H_0^+ + (1,0)$ then $W_{2n} \subset W_n + (1,0) \subset H^+ + (2,0)$
  - $\implies W_{jn} \subset H_0^+ + (j,0)$ for all $j \in \mathbb{N}$
- $\partial W_{jn} \subset \bigcup_{k=-jn}^{jn} \widetilde{f}^k(\partial H_0^+) \subset H_0^+ + (j,0)$;
- $\implies \exists\ z_j = (0, y_j),\ k_j \in \mathbb{Z}$ with $|k_j| \leqslant jn$, $(\widetilde{f}^{k_j}(z_j))_1 \geqslant j$.

$$\left|\frac{(\widetilde{f}^{k_j}(z_j) - z_j)_1}{k_j}\right| \geqslant \left|\frac{j}{k_j}\right| \geqslant \frac{1}{n}. \quad \text{not possible!}$$

So $W_n \cap [0,1]^2 \neq \varnothing$ for all $n \implies \omega_0^+ \neq \varnothing$.

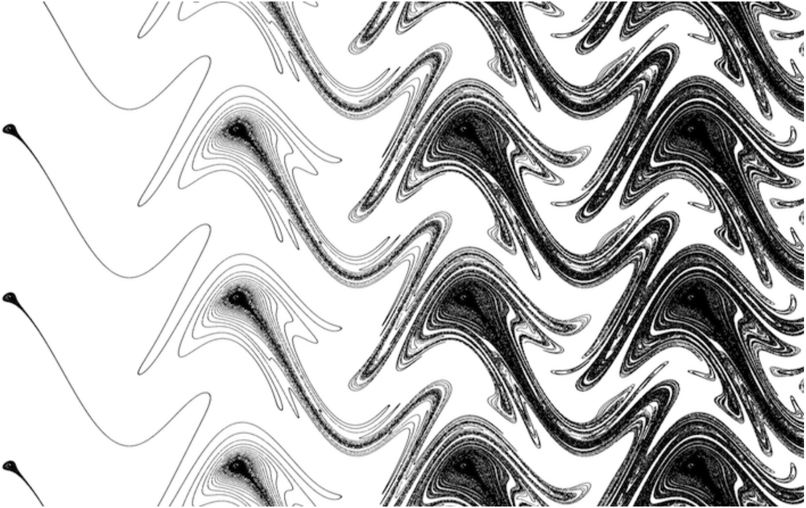
Remark

$H_s^{\pm} \subset \omega_0^{\pm}$ for some $s \implies \widetilde{f}$ has bounded horizontal displacement.

Thus if we assume that $\rho(\widetilde{f}) = \{0\} \times [a, b]$ and $\widetilde{f}$ does NOT have uniformly bounded horizontal displacement:

- ω_0^+ and ω_0^- are nonempty;
- They do not contain a half-plane;
- $\mathbb{R}^2 \backslash \omega_0^+$ is unbounded to the right, $\mathbb{R}^2 \backslash \omega_0^-$ is unbounded to the left.

ω-sets: example



Vertical rotation interval

Proof of:

$\rho(\widetilde{f}) = \{0\} \times [a,b] \implies$ uniformly bounded horizontal displacement.

Suppose horizontal displacement is not uniformly bounded.

- f has strictly toral dynamics; $\mathcal{C}(f) = \mathrm{Ess}(f)$ is fully essential.
- We may assume $\{0\} \times [-1,1] \subset \rho(\widetilde{f})$ (replace $\widetilde{f}$ by $\widetilde{f}^n - (0,m)$).
- There are points z_i such that $\widetilde{f}(z_i) = z_i + (0,i)$, $i \in \{-1,0,1\}$.
- These points can be chosen in $\mathcal{C}(f)$.
- ω_0^+ and ω_0^- are nonempty.
- $\mathcal{C}(f) \subset \overline{\pi(\omega_0^+)} \cap \overline{\pi(\omega_0^-)}$.
- Fact: $\omega_k^- \cap \omega_0^+ \neq \emptyset$ for some $k \in \mathbb{Z}$.
- Fact (difficult): There is $k \in \mathbb{Z}$ such that $\{z_{-1}, z_0, z_1\} \subset \omega_k^-$.

Strictly toral dynamics

Theorem [KT14]

One of the following holds:

(1) $\exists n > 0$, $\mathrm{Fix}(f^n)$ is essential;

(2) $\exists n > 0$, f^n is "annular" ($\exists v \in \mathbb{Z}^2_*$, $\langle \Delta^n_{\tilde{f}}(z), v\rangle$ bounded).

(3) $\mathrm{Ine}(f)$ is a disjoint union of periodic homotopically bounded topdisks, $\mathrm{Ess}(f)$ is a fully essential continuum and $\mathcal{C}(f) = \mathrm{Ess}(f)$. Moreover:
- $\mathcal{C}(f)$ is weakly syndetically transitive;
- For any lift $\tilde{f}$ of f and U neighborhood of $x \in \mathcal{C}(f)$, $\rho(\tilde{f}, U) = \rho(\tilde{f})$.
- Every rotation vector realized by a periodic point or ergodic measure can be realized in $\mathcal{C}(f)$.

If (1) or (2) holds, we may think of f as a "reducible" map. Otherwise we say f is **strictly toral**.

Dynamics in the cylinder

- $\mathbb{A} = \mathbb{R}^2/\langle T^2\rangle$ where $T\colon (x,y) \mapsto (x, y+1)$
- $\tau\colon \mathbb{R}^2 \to \mathbb{A}$ projection;
- $\widehat{f}\colon \mathbb{A} \to \mathbb{A}$ induced by $\widetilde{f}$;
- **Claim:** $\widehat{f}$ **has no wandering points**.
- $W = \tau(\omega_k^-)$ contains $\widehat{z}_i = \tau(z_i)$, $i \in \{-1, 0, 1\}$.
- Compactify with topological ends: $\mathbb{S}^2 = \mathbb{A} \cup \{\pm\infty\}$.
- $U = \mathbb{S}^2 \backslash (W \cup \{-\infty\})$ is an open $\widehat{f}$-invariant topological disk.

Theorem [KLCN15]

If $h \in \mathrm{Homeo}_+(\mathbb{S}^2)$ leaves invariant an open topdisk U without wandering points and the prime ends rotation number in U is nonzero, then there is at most one fixed point in $\mathbb{S}^2 \backslash U$.

Replace $\widetilde{f}$ by $T\widetilde{f}$ if needed to guarantee nonzero rotation number.
But there is a fixed point in W ($\widehat{z}_0$ or $\widehat{z}_{-1}$) besides $-\infty$. Contradiction.

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