Rotation theory of torus homeomorphisms III

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Area-preserving homeomorphisms

From now on, f preserves lebesgue measure μ.

Remark

In the topological setting, by the Oxtoby-Ulam theorem this is equivalent to saying that f preserves a non-atomic measure of full support: for any such measure ν there exists a homeomorphism h such that $h_*(\nu) = \mu$.

The mean rotation vector $\rho(\widetilde{f}, \mu) = \int \Delta_{\widetilde{f}} d\mu$ is particularly useful.

- $\rho(\widetilde{f}, \mu) = (0,0) \implies \mathrm{Fix}(\widetilde{f}) \neq \varnothing$ [Fra88, LC97];
- In particular, if $\rho(\widetilde{f}, \mu)$ is rational, it is realized by a periodic point;
- $\widetilde{f} \in \widetilde{\mathrm{Diff}}^r_{0,\mu}(\mathbb{T}^2) \mapsto \rho(\widetilde{f}, \mu) \in \mathbb{R}^2$ is a group homomorphism (exercise);
- In particular, it is easy to perturb: $\rho(\widetilde{f} + v) = \rho(\widetilde{f}) + v \ \forall v \in \mathbb{R}^2$;
- C^r-generically in $\mathrm{Diff}^r_{0,\mu}(\mathbb{T}^2)$ there are periodic points;
- C^r-generically in $\mathrm{Diff}^r_{0,\mu}(\mathbb{T}^2)$, the rotation set has nonempty interior.

Irrotational area-preserving homeomorphisms

Theorem (Lifted Poincaré recurrence [KT14b])

If f is area-preserving and irrotational (i.e. $\rho(\widetilde{f}) = \{(0,0)\}$), then almost every $z \in \mathbb{R}^2$ is $\widetilde{f}$-recurrent.

Theorem [KT14b, Tal15, LCT15]

If f is area-preserving and irrotational, then either $\mathrm{Fix}(f)$ is essential or the displacement is uniformly bounded: $\sup_{z\in\mathbb{R}^2, n\in\mathbb{Z}} \|\widetilde{f}^n(z) - z\| < \infty$.

Question

Does the recurrence on the lift hold for irrotational area-preserving homeomorphisms on arbitrary surfaces?

Irrotational example

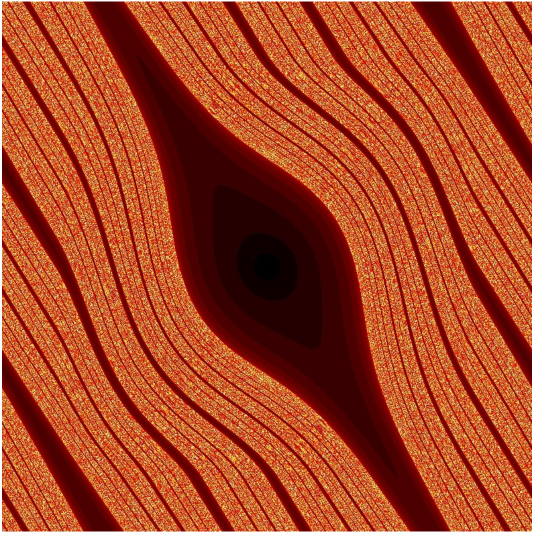
Irrotational diffeomorphisms with unbounded deviations [KT14a]

There exists a C^∞ Bernoulli ($\Longrightarrow$ ergodic) diffeomorphism f with a lift $\widetilde{f}$ such that $\rho(\widetilde{f}) = \{(0,0)\}$ and the displacement is unbounded in all directions. More specifically, the orbit of almost every point intersects every fundamental domain in $\mathbb{R}^2$.

- Find an open topological disk U in $\mathbb{T}^2$ in a way that its lift to $\mathbb{R}^2$ intersects every fundamental domain.
- Choose a smooth ergodic diffeomorphism ϕ of the unit disk $\mathbb{D}$ which is the identity on $\partial\,\mathbb{D}$ and $\phi - \mathrm{Id}$ goes to 0 sufficiently fast near $\partial\,\mathbb{D}$ (Katok 1979).
- Extend as the identity on $\mathbb{T}^2 \backslash U$.
- Simpler example: blow up an orbit of a minimal flow on $\mathbb{T}^2$.

Note: $\mathrm{Fix}(f)$ **is huge!**.

Unbounded disk (with direction)



Area-preserving homeomorphisms

Two questions

(1) f irrotational + unbounded displacement $\implies$ huge fixed point set? $(\mathbb{T}^2 \setminus \mathrm{Fix}(f) \subset \cup\{\text{invariant disks}\})$

(2) Unbounded invariant topological disks $\implies$ huge fixed point set?

Yes!

- (1):[KT14b, LCT15].
- (2): [KT14c, KT15].

General philosophy

If an open connected set U is invariant by an area-preserving homeomorphism, there are strong restrictions on the topology of ∂U (unless f has a "huge" set of fixed points).

In the area-preserving setting, connected open invariant (periodic) sets appear frequently: if U is open, the connected component of U in $\mathcal{O}_f(U) = \bigcup_{n\in\mathbb{Z}} f^n(U)$ is periodic. Also: KAM.

Bounded disks lemma

Recall: U inessential $\iff$ every loop in U is trivial in $\mathbb{T}^2$. An arbitrary set is inessential if it has an inessential neighborhood.

Covering diameter

For U open connected and inessential, $\mathcal{D}(U) = \mathrm{diam}(\widehat{U})$ where $\widehat{U}$ is a lift of U ($=$ connected component of $\pi^{-1}(U)$).

Bounded disks lemma [KT14c, KT15]

Suppose that f is area-preserving and $\mathrm{Fix}(f)$ is inessential. There exists $M > 0$ such that for any inessential open invariant connected set U one has $\mathcal{D}(U) \leqslant M$.

It holds on any surface. There is a version for non-simply connected sets.

Application: dynamically essential and inessential points

An open set $U \subset \mathbb{T}^2$ is **fully essential** in $\mathbb{T}^2$ if $\mathbb{T}^2 \setminus U$ is inessential.

Dynamically essential/inessential points

- $x \in \mathrm{Ine}(f) =$ **dynamically inessential** points if there is a neighborhood U of x such that $\mathcal{O}_f(U)$ is inessential in $\mathbb{T}^2$.
- $x \in \mathrm{Ess}(f) =$ **dynamically essential** points if $\mathcal{O}_f(U)$ is essential for every neighborhood U of x.
- $x \in \mathcal{C}(f) =$ **dynamically fully essential** points if $\mathcal{O}_f(U)$ is fully essential for every neighborhood U of x.

Area preserving $\implies$ every $x \in \mathrm{Ine}(f)$ belongs to a periodic open topdisk.

- $\mathrm{Ine}(f)$ is open invariant;
- $\mathrm{Ess}(f) = \mathbb{T}^2 \setminus \mathrm{Ine}(f)$ and $\mathcal{C}(f)$ are compact invariant.

Note: $\mathrm{Ine}(f)$ may be essential as a set, $\mathrm{Ess}(f)$ may be inessential.

Strictly toral dynamics

Theorem [KT14c]

One of the following holds:

(1) $\exists n > 0$, $\mathrm{Fix}(f^n)$ is essential;

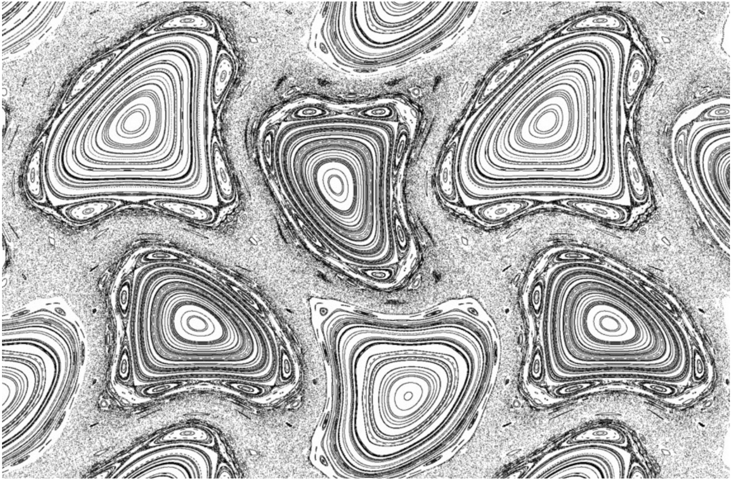
(2) $\exists n > 0$, f^n is "annular" ($\exists v \in \mathbb{Z}^2_*$, $\langle \Delta^n_{\tilde{f}}(z), v\rangle$ bounded).

(3) $\mathrm{Ine}(f)$ is a disjoint union of periodic homotopically bounded topdisks, $\mathrm{Ess}(f)$ is a fully essential continuum and $\mathcal{C}(f) = \mathrm{Ess}(f)$. Moreover:
  - $\mathcal{C}(f)$ is weakly syndetically transitive;
  - For any lift $\widetilde{f}$ of f and U neighborhood of $x \in \mathcal{C}(f)$, $\rho(\widetilde{f}, U) = \rho(\widetilde{f})$.
  - Every rotation vector realized by a periodic point or ergodic measure can be realized in $\mathcal{C}(f)$.

If (1) or (2) holds, we may think of f as a "reducible" map. Otherwise we say f is **strictly toral**.

Example: $\operatorname{int} \rho(\widetilde{f}) \neq \varnothing \implies$ strictly toral (exercise).
There is a version for higher genus surfaces [KT15].

Strictly toral dynamics



Proof of theorem using BDL

Blackboard

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