Rotation theory of torus homeomorphisms III

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Area-preserving homeomorphisms

From now on, \( f \) preserves Lebesgue measure \( \mu \).

**Remark**

In the topological setting, by the Oxtoby-Ulam theorem this is equivalent to saying that \( f \) preserves a non-atomic measure of full support: for any such measure \( \nu \) there exists a homeomorphism \( h \) such that \( h_*(\nu) = \mu \).

The mean rotation vector \( \rho(\tilde{f}, \mu) = \int \Delta \tilde{f} d\mu \) is particularly useful.

- \( \rho(\tilde{f}, \mu) = (0, 0) \implies \text{Fix}(\tilde{f}) \neq \emptyset \) [Fra88, LC97];
- In particular, if \( \rho(\tilde{f}, \mu) \) is rational, it is realized by a periodic point;
- \( \tilde{f} \in \tilde{\text{Diff}}_{0,\mu}(\mathbb{T}^2) \mapsto \rho(\tilde{f}, \mu) \in \mathbb{R}^2 \) is a group homomorphism (exercise);
- In particular, it is easy to perturb: \( \rho(\tilde{f} + v) = \rho(\tilde{f}) + v \) \( \forall v \in \mathbb{R}^2 \);
- \( C^r \)-generically in \( \text{Diff}_{0,\mu}^r(\mathbb{T}^2) \) there are periodic points;
- \( C^r \)-generically in \( \text{Diff}_{0,\mu}^r(\mathbb{T}^2) \), the rotation set has nonempty interior.
Irrotational area-preserving homeomorphisms

**Theorem (Lifted Poincaré recurrence [KT14b])**

If $f$ is area-preserving and irrotational (i.e. $\rho(\tilde{f}) = \{(0, 0)\})$, then almost every $z \in \mathbb{R}^2$ is $\tilde{f}$-recurrent.

**Theorem [KT14b, Tal15, LCT15]**

If $f$ is area-preserving and irrotational, then either $\text{Fix}(f)$ is essential or the displacement is uniformly bounded: $\sup_{z \in \mathbb{R}^2, n \in \mathbb{Z}} \|\tilde{f}^n(z) - z\| < \infty$.

**Question**

Does the recurrence on the lift hold for irrotational area-preserving homeomorphisms on arbitrary surfaces?
Irrotational example

Irrotational diffeomorphisms with unbounded deviations [KT14a]

There exists a $C^\infty$ Bernoulli (ergodic) diffeomorphism $f$ with a lift $\tilde{f}$ such that $\rho(\tilde{f}) = \{(0,0)\}$ and the displacement is unbounded in all directions. More specifically, the orbit of almost every point intersects every fundamental domain in $\mathbb{R}^2$.

- Find an open topological disk $U$ in $\mathbb{T}^2$ in a way that its lift to $\mathbb{R}^2$ intersects every fundamental domain.
- Choose a smooth ergodic diffeomorphism $\phi$ of the unit disk $\mathbb{D}$ which is the identity on $\partial \mathbb{D}$ and $\phi - \text{Id}$ goes to 0 sufficiently fast near $\partial \mathbb{D}$ (Katok 1979).
- Extend as the identity on $\mathbb{T}^2 \setminus U$.
- Simpler example: blow up an orbit of a minimal flow on $\mathbb{T}^2$.

Note: $\text{Fix}(f)$ is huge!
Unbounded disk (with direction)
Area-preserving homeomorphisms

Two questions

(1) \( f \) irrotational + unbounded displacement \( \Rightarrow \) huge fixed point set?
   \( (\mathbb{T}^2 \setminus \text{Fix}(f) \cup \{\text{invariant disks}\}) \)

(2) Unbounded invariant topological disks \( \Rightarrow \) huge fixed point set?

Yes!

- (1): [KT14b, LCT15].
- (2): [KT14c, KT15].

General philosophy

If an open connected set \( U \) is invariant by an area-preserving homeomorphism, there are strong restrictions on the topology of \( \partial U \) (unless \( f \) has a “huge” set of fixed points).

In the area-preserving setting, connected open invariant (periodic) sets appear frequently: if \( U \) is open, the connected component of \( U \) in \( \mathcal{O}_f(U) = \bigcup_{n \in \mathbb{Z}} f^n(U) \) is periodic. Also: KAM.
Bounded disks lemma

Recall: \( U \) inessential \iff every loop in \( U \) is trivial in \( \mathbb{T}^2 \). An arbitrary set is inessential if it has an inessential neighborhood.

**Covering diameter**

For \( U \) open connected and inessential, \( D(U) = \text{diam} (\hat{U}) \) where \( \hat{U} \) is a lift of \( U \) (\( = \) connected component of \( \pi^{-1}(U) \)).

**Bounded disks lemma [KT14c, KT15]**

Suppose that \( f \) is area-preserving and \( \text{Fix}(f) \) is inessential. There exists \( M > 0 \) such that for any inessential open invariant connected set \( U \) one has \( D(U) \leq M \).

It holds on any surface. There is a version for non-simply connected sets.
Application: dynamically essential and inessential points

An open set $U \subset \mathbb{T}^2$ is **fully essential** in $\mathbb{T}^2$ if $\mathbb{T}^2 \setminus U$ is inessential.

**Dynamically essential/inessential points**

- $x \in \text{Ine}(f) = \text{dynamically inessential}$ points if there is a neighborhood $U$ of $x$ such that $O_f(U)$ is inessential in $\mathbb{T}^2$.
- $x \in \text{Ess}(f) = \text{dynamically essential}$ points if $O_f(U)$ is essential for every neighborhood $U$ of $x$.
- $x \in \mathcal{C}(f) = \text{dynamically fully essential}$ points if $O_f(U)$ is fully essential for every neighborhood $U$ of $x$.

Area preserving $\implies$ every $x \in \text{Ine}(f)$ belongs to a periodic open topdisk.

- $\text{Ine}(f)$ is open invariant;
- $\text{Ess}(f) = \mathbb{T}^2 \setminus \text{Ine}(f)$ and $\mathcal{C}(f)$ are compact invariant.

Note: $\text{Ine}(f)$ may be essential as a set, $\text{Ess}(f)$ may be inessential.
Strictly toral dynamics

**Theorem [KT14c]**

One of the following holds:

(1) $\exists n > 0$, $\text{Fix}(f^n)$ is essential;

(2) $\exists n > 0$, $f^n$ is “annular” ($\exists v \in \mathbb{Z}_*^2$, $\langle \Delta_{f^n}(z), v \rangle$ bounded).

(3) $\text{Ine}(f)$ is a disjoint union of periodic homotopically bounded topdisks, $\text{Ess}(f)$ is a fully essential continuum and $C(f) = \text{Ess}(f)$. Moreover:
   - $C(f)$ is weakly syndetically transitive;
   - For any lift $\tilde{f}$ of $f$ and $U$ neighborhood of $x \in C(f)$, $\rho(\tilde{f}, U) = \rho(\tilde{f})$.
   - Every rotation vector realized by a periodic point or ergodic measure can be realized in $C(f)$.

If (1) or (2) holds, we may think of $f$ as a “reducible” map. Otherwise we say $f$ is **strictly toral**.

Example: $\text{int} \rho(\tilde{f}) \neq \emptyset \implies$ strictly toral (exercise).

There is a version for higher genus surfaces [KT15].
Strictly toral dynamics
Proof of theorem using BDL

Blackboard
References


