Rotation theory of torus homeomorphisms II

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Setting

- $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$; $\pi\colon \mathbb{R}^2 \to \mathbb{T}^2$ projection;
- $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$, $\widetilde{f}\colon \mathbb{R}^2 \to \mathbb{R}^2$ a lift of f;
- $\widetilde{f}(z+v) = \widetilde{f}(z) + v$, $\forall v \in \mathbb{Z}^2$;
- Displacement function $\widetilde{\Delta}_{\widetilde{f}} = \widetilde{f} - \mathrm{Id}\colon \mathbb{R}^2 \to \mathbb{R}^2$ ($\mathbb{Z}^2$-periodic);
- Induces $\Delta_{\widetilde{f}}\colon \mathbb{T}^2 \to \mathbb{R}^2$, $\Delta_{\widetilde{f}}(z) = \widetilde{\Delta}_{\widetilde{f}}(\widetilde{z})$, for $\widetilde{z} \in \pi^{-1}(z)$;
- Notation: $\Delta^n_{\widetilde{f}} = \sum_{k=0}^{n-1} \Delta_{\widetilde{f}} \circ f^k$ (displacement cocycle).

Rotation sets of toral homeomorphisms

- $z \in \mathbb{T}^2$, $\widetilde{z} \in \pi^{-1}(z)$. **Rotation vector of** z (if the limit exists):

$$\rho(\widetilde{f}, z) = \lim_{n\to\infty} \frac{\widetilde{f}^n(\widetilde{z}) - \widetilde{z}}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \Delta_f(f^n(z)) = \lim_{n\to\infty} \frac{1}{n} \Delta_{\widetilde{f}}^n(z)$$

- Birkhoff: if $\mu \in \mathcal{M}(f)$, then $\rho(\widetilde{f}, z)$ exists for μ-a.e. z and

$$\int \rho(\widetilde{f}, z)\, d\mu(z) = \int \Delta_{\widetilde{f}}\, d\mu := \rho(\widetilde{f}, \mu) \text{ (\textbf{mean rotation vector for} } \mu)$$

- If μ ergodic, $\rho(\widetilde{f}, z) = \rho(\widetilde{f}, \mu)$ for μ-a.e. z.
- **Measure rotation set**: $\rho_m(\widetilde{f}) = \{\rho(\widetilde{f}, \mu) : \mu \in \mathcal{M}(f)\}$.
- Compact and convex. $\mathrm{Ext}(\rho_m(\widetilde{f})) \subset \rho_{\mathsf{erg}}(\widetilde{f}) := \{\rho(\widetilde{f}, \mu) : \mu \text{ ergodic}\}$.
- **Pointwise rotation set**: $\rho_p(\widetilde{f}) = \{\text{rotation vectors of points}\}$.
- $\rho_m(\widetilde{f}) \supset \rho_p(\widetilde{f}) \supset \rho_{\mathsf{erg}}(\widetilde{f}) := \{\rho(\widetilde{f}, \mu) : \mu \text{ ergodic}\}$..

Rotation sets of toral homeomorphisms

Pointwise: difficult to work with. Invariant measures: too weak.

Misiurewicz-Ziemian rotation set [MZ89]

$$\begin{aligned}\rho(\widetilde{f}) &= \left\{ \lim_{k\to\infty} (\widetilde{f}^{n_k}(z_k) - z_k)/n_k : z_k \in \mathbb{R}^2,\, n_k \to \infty \right\} \\ &= \left\{ \lim_{k\to\infty} \frac{1}{n_k} \Delta_{\widetilde{f}}^{n_k}(z_k) : z_k \in \mathbb{T}^2,\, n_k \to \infty \right\} \\ &= \bigcap_{n\geqslant 0} \overline{\bigcup_{m\geqslant n} \frac{1}{m} \Delta_{\widetilde{f}}^{m}(\mathbb{T}^2)} = \bigcap_{n\geqslant 0} \overline{\bigcup_{m\geqslant n} \frac{1}{m} \widetilde{\Delta}_{\widetilde{f}}^{m}([0,1]^2)}\end{aligned}$$

Proposition

$\rho_{\mathsf{erg}}(\widetilde{f}) \subset \rho_p(\widetilde{f}) \subset \rho(\widetilde{f}) = \rho_m(\widetilde{f})$ (dimension 2!)

Proof: $\rho_m(\widetilde{f}) = \mathrm{Conv}(\rho_{\mathsf{erg}}(\widetilde{f}))$ and $\rho(\widetilde{f})$ is convex $\implies \rho(\widetilde{f}) \supset \rho_m(\widetilde{f})$.

Convexity of the Misiurewicz-Ziemian rotation set

$$Q = (0,1)^2 \text{ unit square}, \quad \rho(\widetilde{f}) = \bigcap_{n\geqslant 0} \overline{\bigcup_{m\geqslant n} \frac{1}{m}\widetilde{\Delta}^m_{\widetilde{f}}(Q)}$$

Follows from:

$\widetilde{f}^m(Q)$ is $\sqrt{2}$-**quasi-convex**: $\mathrm{Conv}(\widetilde{f}^m(Q)) \subset B_{\sqrt{2}}(\widetilde{f}^m(Q))$
($\Longrightarrow$ $\frac{1}{m}\widetilde{\Delta}^m_{\widetilde{f}}(Q) = \{(\widetilde{f}^m(z) - z)/m : z \in Q\}$ is $2\sqrt{2}/m$-quasi-convex.)

Lemma (Quasi-convexity)

If $W \subset \mathbb{R}^2$ is open connected and $W \cap (W + v) = \varnothing$ for all $v \in \mathbb{Z}^2_*$, then W is $\sqrt{2}$-quasi-convex, i.e. $\mathrm{Conv}(W) \subset B_{\sqrt{2}}(W)$.

Key lemma (Douady): If γ is a simple arc in $\mathbb{R}^2$ joining x to y and disjoint from the line segment L_{xy} and $v \in \mathbb{R}^2$ is such that $x + v \in D =$ disk bounded by $\gamma \cup L_{xy}$, then $\gamma \cap (\gamma + v) \neq \varnothing$. **(Blackboard.)**

Rotation sets with nonempty interior: periodic points

A rational point $(p_1, p_2)/q \in \mathbb{Q}^2$ is **realized by a periodic point** if there is $z \in \mathbb{R}^2$ such that $\widetilde{f}^q(z) = z + (p_1, p_2)$.

Theorems [Fra89, Fra88]

- Every rational **extremal** point of $\rho(\widetilde{f})$ is realized by a periodic point;
- Every rational point in the **interior** of $\rho(\widetilde{f})$ is also realized.

In particular, $\operatorname{int}(\rho(\widetilde{f})) \neq \varnothing \implies$ infinitely many periodic points.

Theorem [MZ91]

Every element of $\operatorname{int}(\rho(\widetilde{f}))$ is realized by an ergodic measure.

In particular, $\operatorname{int}(\rho(\widetilde{f})) \subset \rho_p(\widetilde{f})$.

Rotation sets with nonempty interior: entropy

Theorem [LM91]

If $\rho(\widetilde{f})$ has nonempty interior, then f has positive topological entropy.

Main tool: Thurston classification, Handel's global shadowing.

Theorem (Nielsen-Thurston) [FLP12]

Let $g\colon S \to S$ be a homeomorphism of a compact surface. Then g is **isotopic** to a homeomorphism Φ of one of these types:

- Periodic: $\Phi^k = \mathrm{Id}$ for some k;
- Pseudo-Anosov:
  - There exist transverse measured foliations $\mathcal{F}^s$, $\mathcal{F}^u$ with (finitely many) singularities and $\lambda > 1$ such that $\Phi(\mathcal{F}^s) = \lambda^{-1}\mathcal{F}^s$, $\Phi(\mathcal{F}^u) = \lambda\mathcal{F}^u$.
- Reducible:
  - There is an invariant finite union of non-peripheral essential simple loops.

Rotation sets with nonempty interior: entropy

Pseudo-Anosov maps

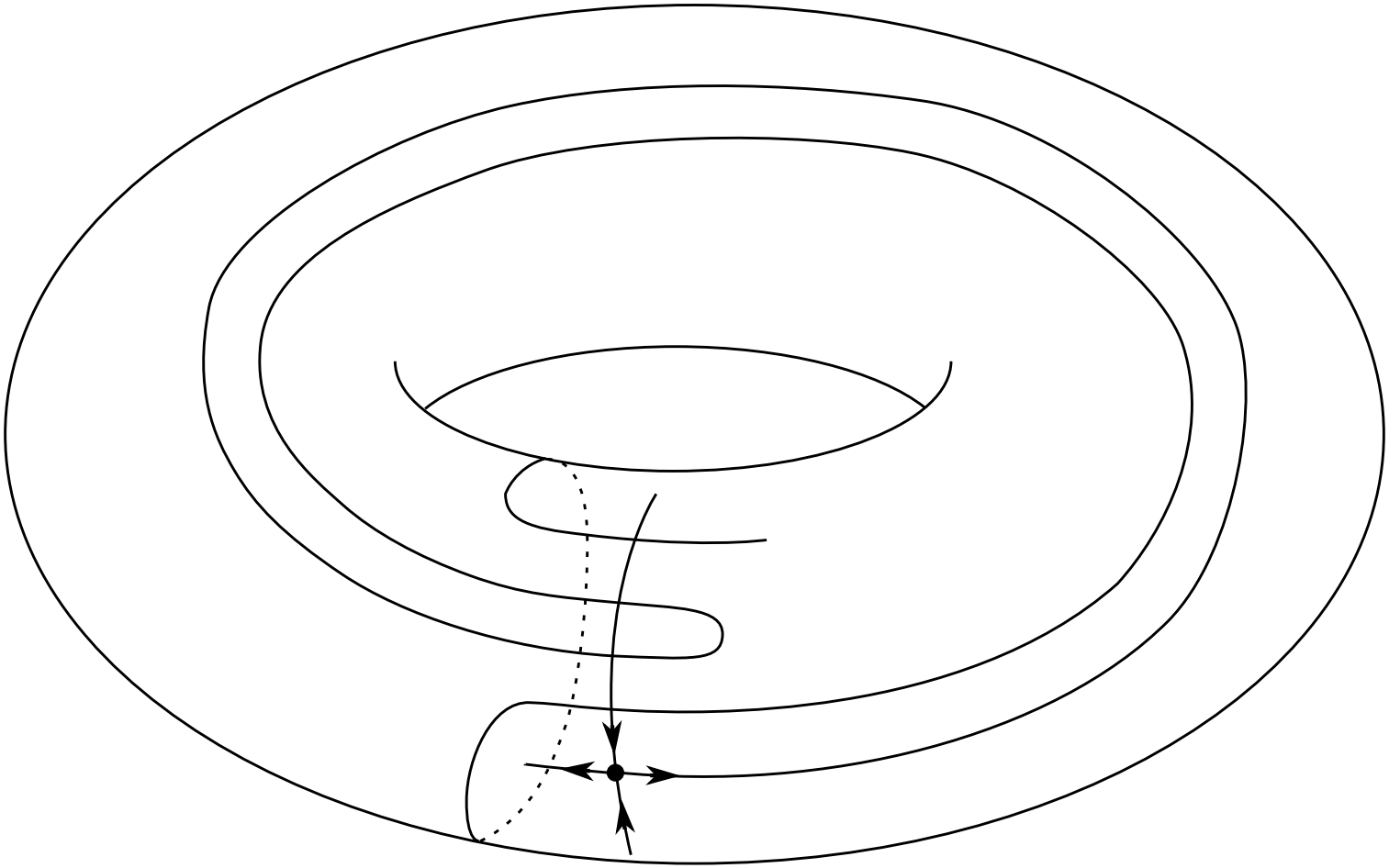
- They are transitive, have dense periodic points.
- Markov partitions, positive entropy.
- Stability properties: g isotopic to Φ (pA-map) $\Longrightarrow$
  - $\exists Y \subset\subset S$ and $h\colon Y \to S$ continuous surjection, $hf|_Y = \Phi h$.
  - $h_{top}(\Phi) \leqslant h_{top}(f)$ [Han85].
  - Periodic points of Φ "persist" by isotopy.

Similar results for surfaces with finitely many punctures (marked points).

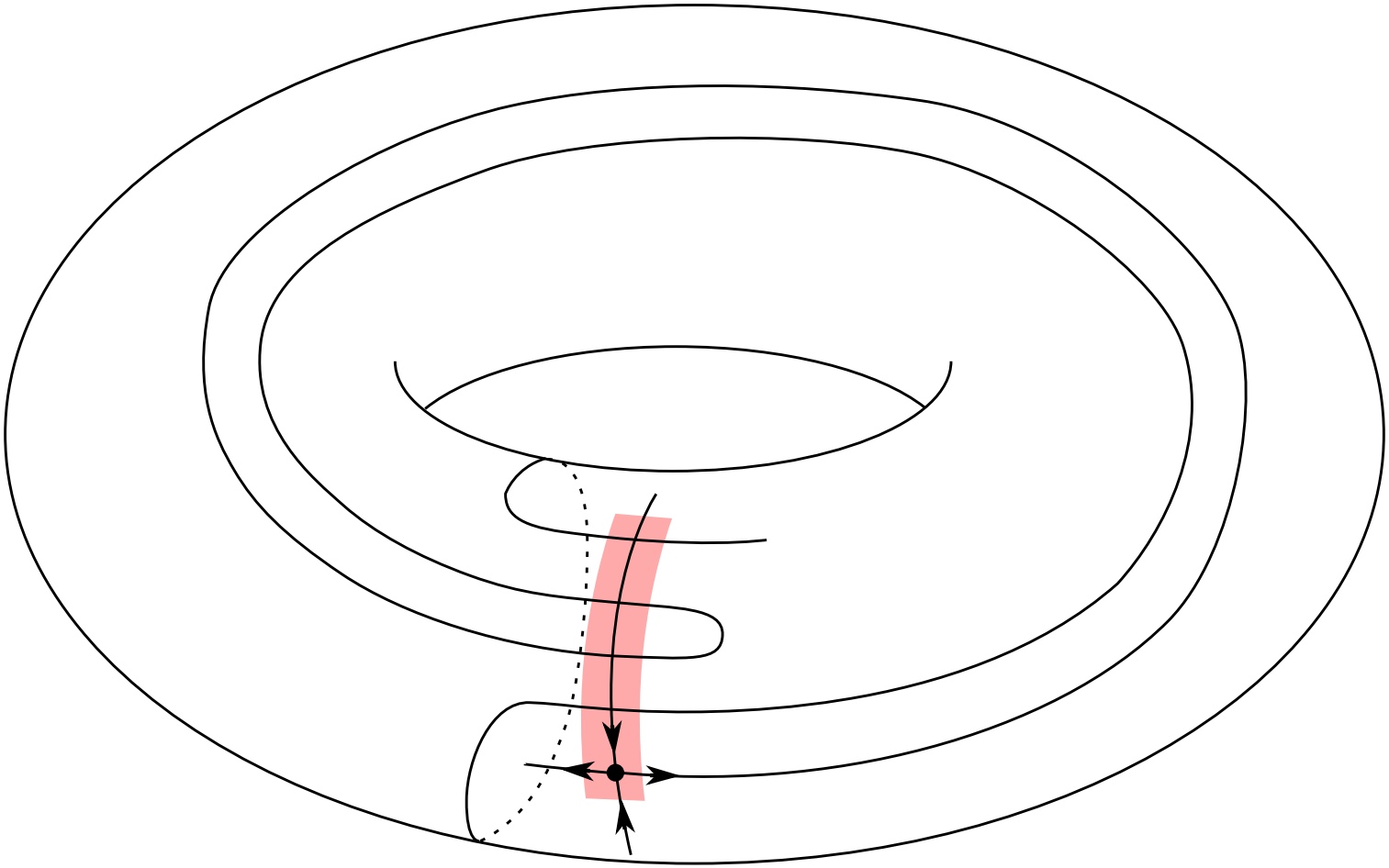
Proof of $\rho(\widetilde{f})$ with interior $\Longrightarrow$ entropy

- Enough to prove it for f^n. We may assume $\rho(\widetilde{f})$ contains a large ball.
- fixed points x_1, x_2, x_3 with non-collinear rotation vectors v_1, v_2, v_3.
- f is isotopic to a (relative) pseudo-Anosov map rel $\{x_1, x_2, x_3\}$.

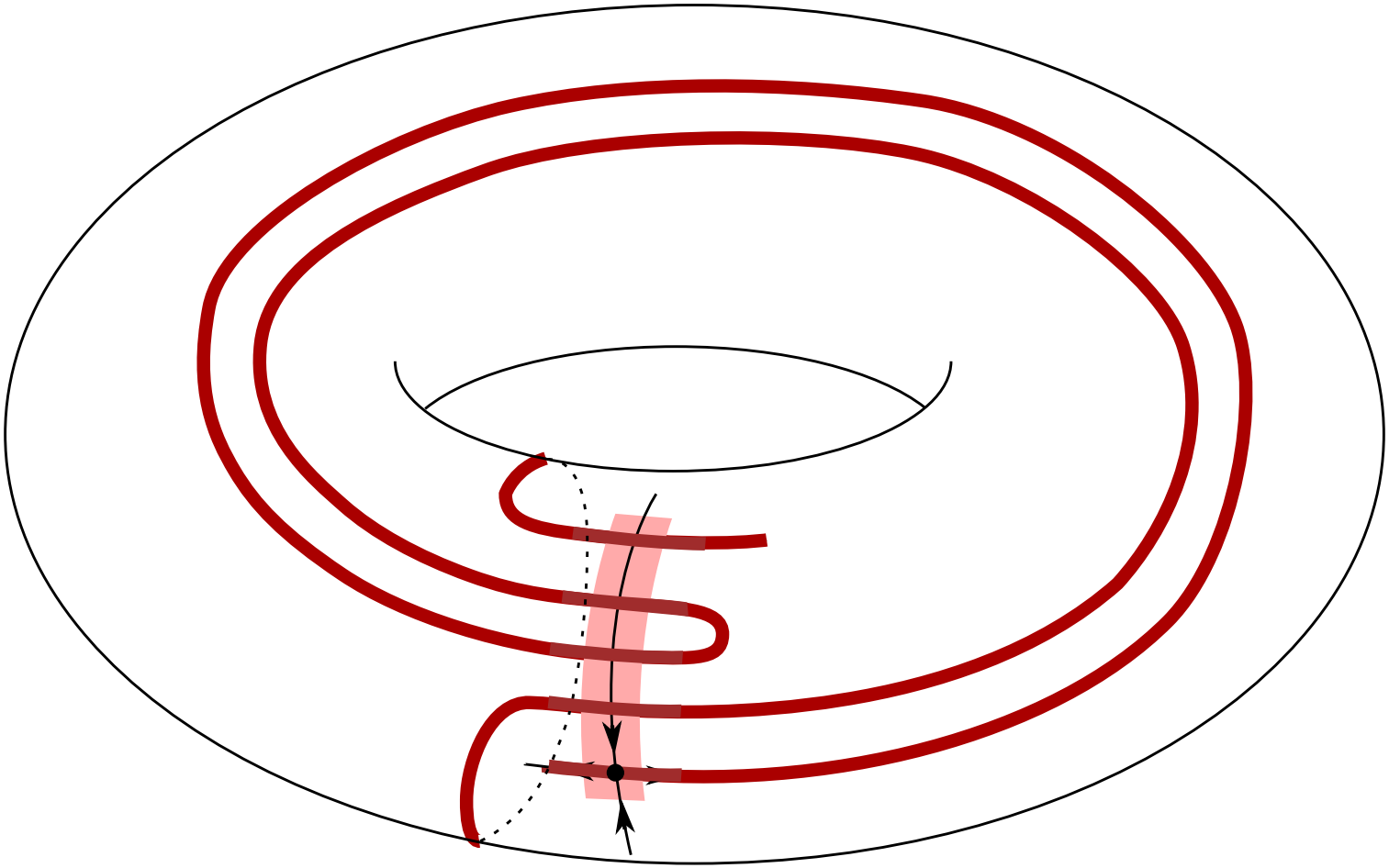
Example



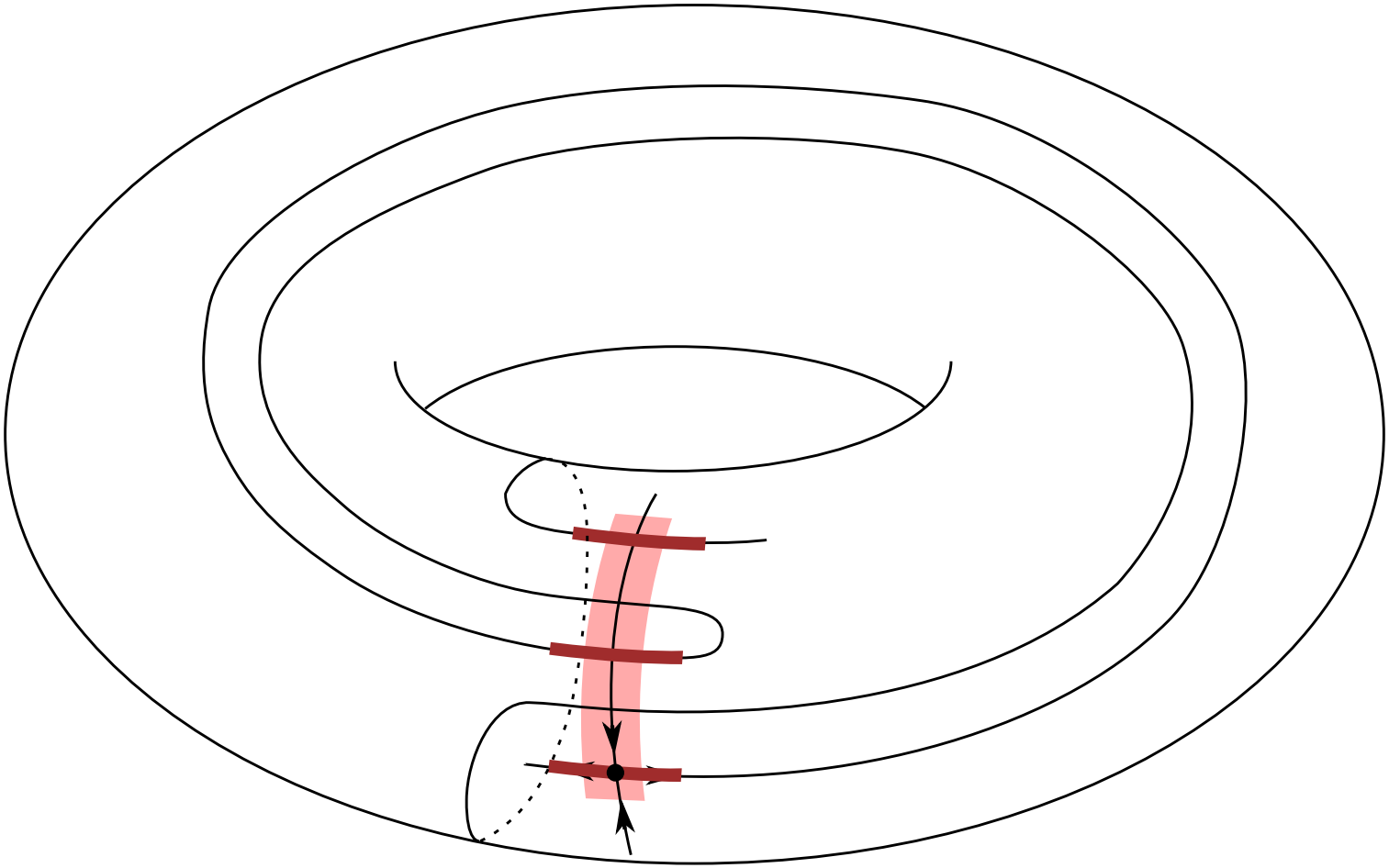
Example

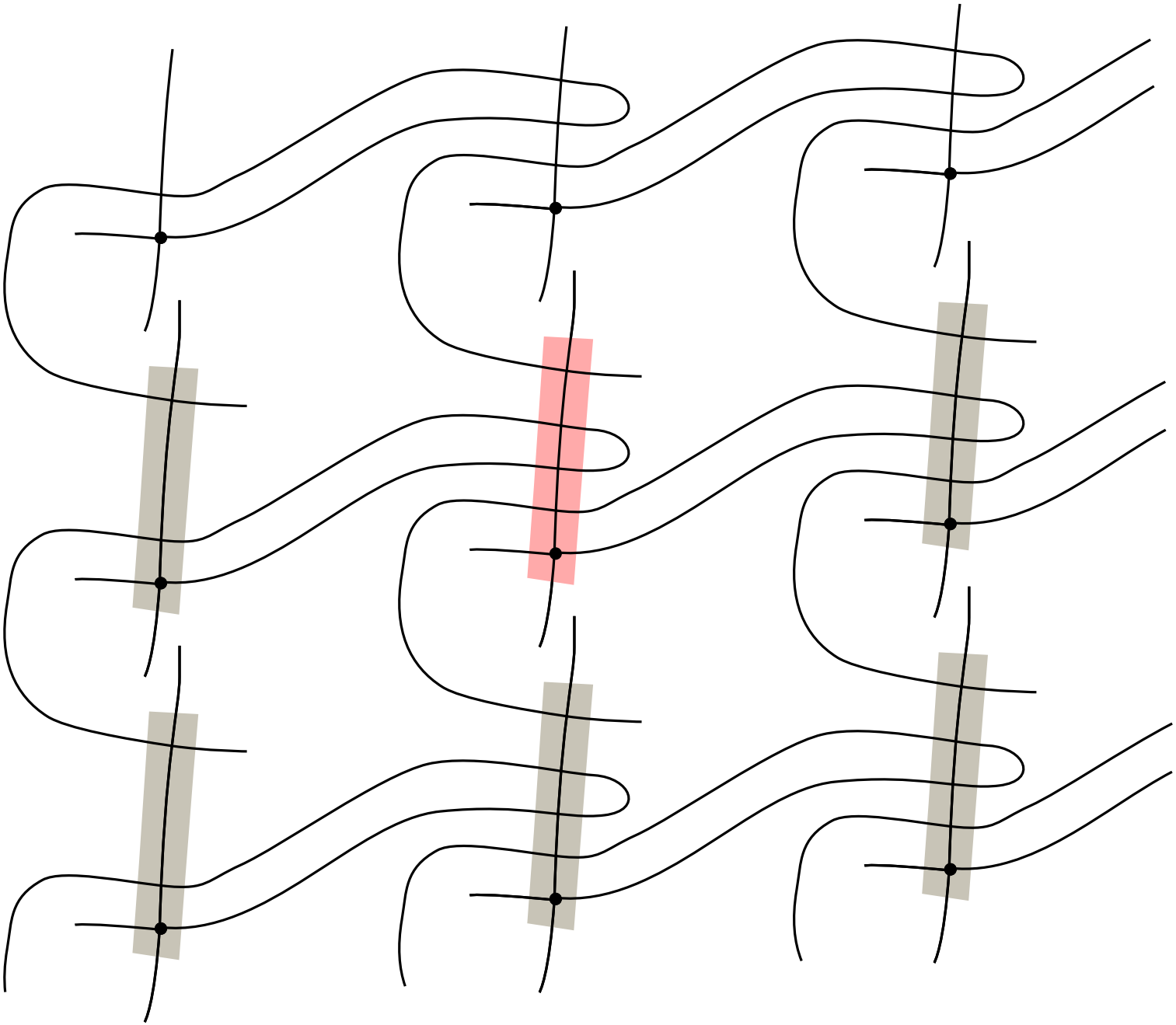


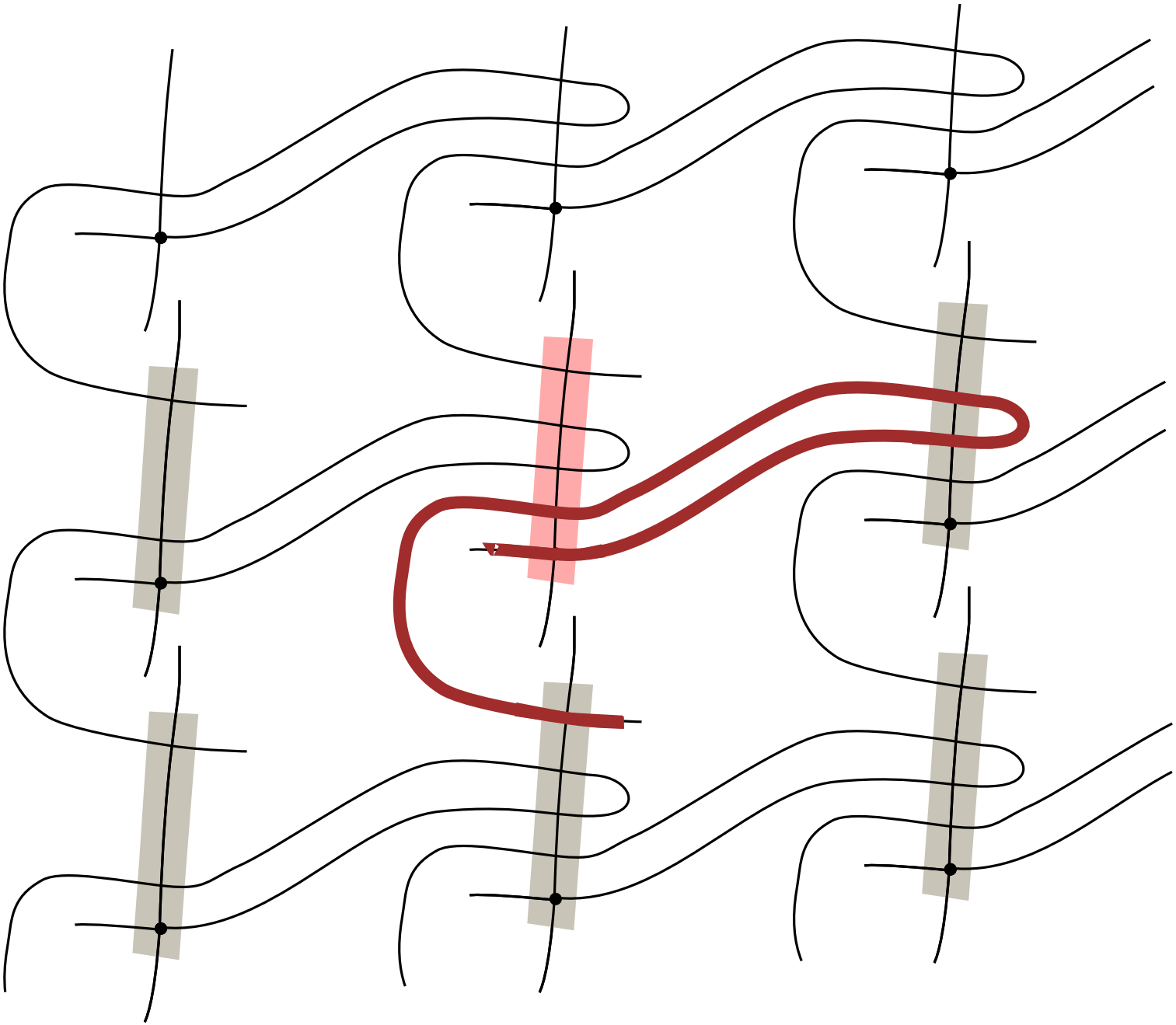
Example

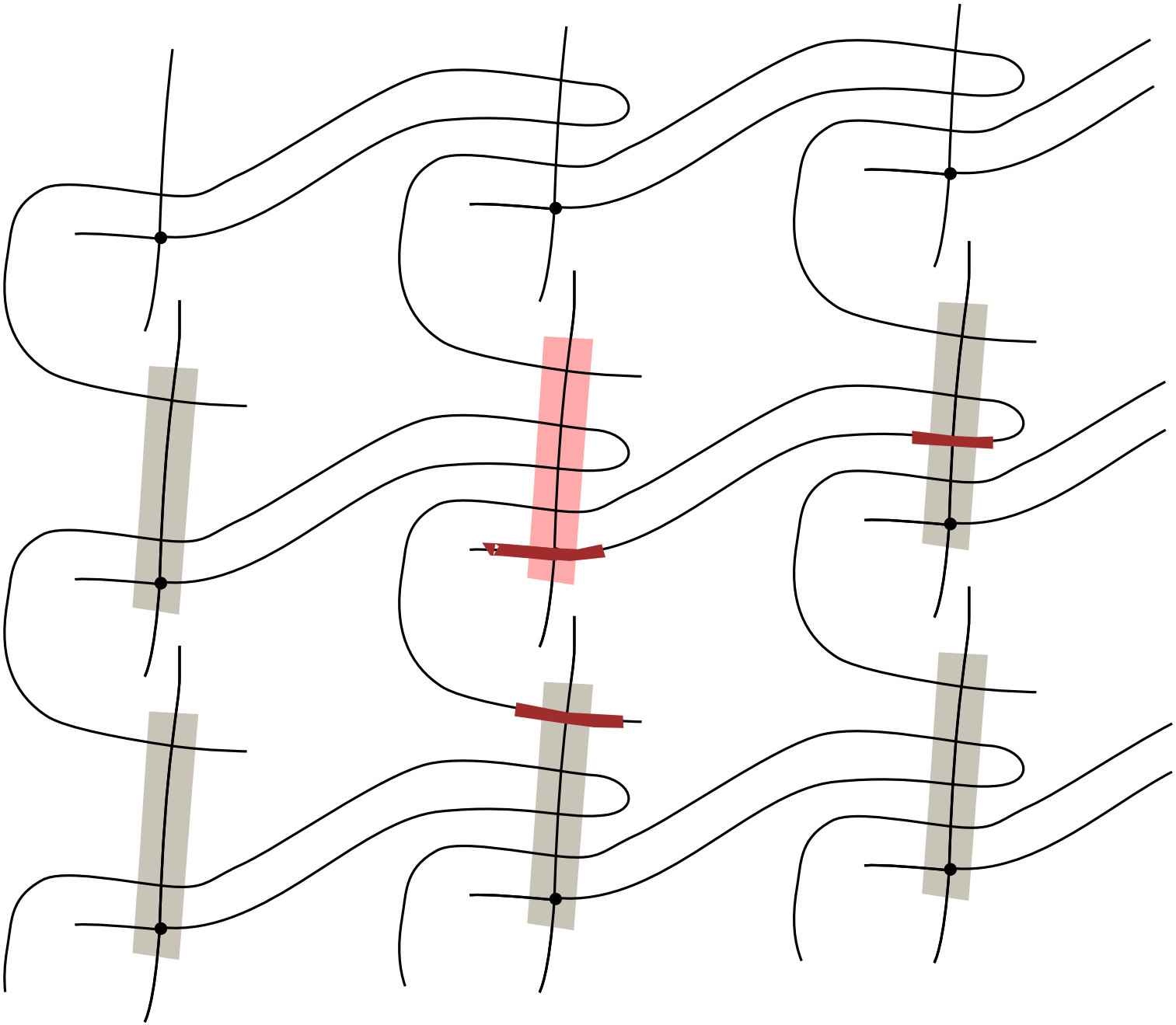


Example









Rotational homoclinic intersections

Theorem [Add15]

If f is $C^{1+\alpha}$ and $(0,0) \in \operatorname{int} \rho(\widetilde{f})$, then there exists a hyperbolic periodic point p for $\widetilde{f}$ such that for all $v \in \mathbb{Z}^2$, the stable manifold of p has a topologically transverse intersection with the unstable manifold of $p+v$ (and vice-versa).

Idea of proof in the case that f is transitive (blackboard).

Theorem [GKT15]

If f is transitive and $(0,0) \in \operatorname{int} \rho(\widetilde{f})$, then $\widetilde{f}$ is transitive.

Restricted rotation sets

The different types of rotation sets can be defined on any subset $K \subset \mathbb{T}^2$ by considering only points on K in the definition. For example

- $\rho_p(\widetilde{f}, K) = \{\text{rotation vectors of points of } K\}$
- $\rho(\widetilde{f}, K) = \{\lim_{k\to\infty} \frac{1}{n_k}\Delta_{\widetilde{f}}^{n_k}(z_k) : z_k \in K,\ n_k \to \infty\}$.

And if K is compact and invariant,

- $\rho_m(\widetilde{f}, K) = \{\rho(\widetilde{f}, \mu) : \mu \in \mathcal{M}(f),\ \mathrm{Supp}(\mu) \subset K\}$
- $\rho_{\mathsf{erg}}(\widetilde{f}, K) = \{\rho(\widetilde{f}, \mu) : \mu \in \mathcal{M}_{\mathsf{erg}}(f),\ \mathrm{Supp}(\mu) \subset K\}$

The latter is always nonempty, and again $\rho_m(\widetilde{f}, K) = \mathrm{Conv}(\rho_{\mathsf{erg}}(\widetilde{f}, K))$.
If $\rho(\widetilde{f}, K) = \{v\}$, we say that K is v-**rotational** ($\iff$ $\rho_p(\widetilde{f}, K) = \{v\}$).

Bounded rotational deviations

A v-rotational K has (uniformly) bounded rotational deviations if $\sup_{z\in K, n\in\mathbb{Z}} \|\Delta_{\widetilde{f}}^n(z) - nv\| < \infty$.

v-rotational sets

Problem

For which values of v can we find v-rotational sets? How 'good'?

Remark [MZ91]

For any compact connected set $C \subset \operatorname{int}(\rho(\widetilde{f}))$ $\exists z$: $\rho(\widetilde{f}, \mathcal{O}(z)) = C$.

Theorems

- For each $v \in \operatorname{int}(\rho(\widetilde{f}))$ there exists a v-rotational set K [MZ91];
- Moreover, it has bounded rotational deviations.
- If v is rational, then one may find $K =$ periodic orbit [Fra89];
- If v is totally irrational, $f|_K$ is semi-conjugate to a rotation [Jäg09].

Rotational deviations

For v in the boundary of $\rho(\widetilde{f})$, there may not exist a v-rotational set. But

Theorem [LCT15]

If $\rho(\widetilde{f})$ has nonempty interior and v is a **vertex** of $\rho(\widetilde{f})$, then for any $\mu \in \mathcal{M}(f)$ with $\rho(\widetilde{f}, \mu) = v$ the support of μ is v-rotational with bounded deviations.

Theorem [BdCH15]

This is not necessarily true if v is only assumed to be an extremal point.

In their article, they define a one-parameter family of homeomorphisms with a rotation set with nonempty interior which undergo a series of bifurcations and they study how the rotation set changes with the parameter. They give very precise descriptions for every parameter and they describe which elements of the rotation set have v-rotational sets.

Rotational deviations

Theorem

If $\rho(\widetilde{f})$ has nonempty interior, then there exists $M > 0$ such that

$$\forall n \in \mathbb{Z}, \quad \{\widetilde{f}^n(z) - z : z \in [0,1]^2\} = \Delta^n_{\widetilde{f}}(\mathbb{T}^2) \subset B_M(n\rho(\widetilde{f})).$$

- Dávalos: rational polygons [Dáv13] (Brouwer-Le Calvez foliations, "forcing")
- Addas-Zanata: $C^{1+\alpha}$ [AZ15] (Pesin theory, homoclinic intersections)
- Tal-Le Calvez: general [LCT15] (Brouwer-Le Calvez foliations, forcing theory)

Rotational deviations: rotation sets with empty interior

Assume $\rho(\widetilde{f})$ has empty interior. Two main cases: interval or point.
If $\rho(\widetilde{f}) = \{v\}$ (i.e. $\mathbb{T}^2$ is v-rotational) we call f a **pseudo-rotation**.

- $v \in \mathbb{R}^2 \backslash \mathbb{Q}^2$ (irrational pseudo-rotation).
  - Dynamics is aperiodc.
  - May be topologically weak-mixing, or even mixing;
  - May have unbounded rotational deviations [KK09];
  - May have positive entropy (but not if f is smooth) [Ree81];
  - All of this may happen for area-preserving maps.
- $v = (p_1/q, p_2/q) \in \mathbb{Q}^2$ (rational pseudo-rotation)
  - Must have periodic points;
  - Interesting case: $v = (0,0)$ (take $\widetilde{g} = \widetilde{f}^q - (p_1, p_2)$).
  - If $\rho(\widetilde{f}) = (0,0)$ we say f is **irrotational**.
  - May have unbounded rotational deviations.
  - Katok's example.
  - Interesting case: f area-preserving.

Irrotational area-preserving homeomorphisms

Theorem (Lifted Poincaré recurrence) [KT14]

If f is area-preserving and irrotational, then a.e. $z \in \mathbb{R}^2$ is $\widetilde{f}$-recurrent.

An open set $U \subset \mathbb{T}^2$ is **essential** it contains a loop homotopically nontrivial in $\mathbb{T}^2$. An arbitrary set is essential if every neighborhood is essential.

Theorem [KT14, Tal15, LCT15]

If f is area-preserving and irrotational, then either $\mathrm{Fix}(f)$ is essential or the displacement is uniformly bounded: $\sup_{z\in\mathbb{T}^2, n\in\mathbb{Z}} \Delta^n_{\widetilde{f}}(z) < \infty$.

Corollary

For an area-preserving rational pseudo-rotation, either $\mathrm{Fix}(f^n)$ is essential or f has uniformly bounded rotational deviations.

Question: Does the lifted Poincaré recurrence hold for arbitrary surfaces?

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